Endogenous Ranking and Equilibrium Lorenz Curve Across (ex-ante) Identical Countries: **A Generalization**

By Kiminori Matsuyama
Northwestern University

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1. Introduction:

- Rich countries tend to have higher TFPs & *K/L* than the poor, typically interpreted as the ***causality*** from TFPs and/or *K/L* to *Y/L*, often under the maintained hypotheses
  - These countries offer independent observations
  - Cross-country variations would disappear without any exogenous variations.

- A complementary approach in ***trade*** (and ***economic geography***): even if countries are ex-ante identical, interaction through trade (and factor mobility) could lead to:
  - Equilibrium dispersions in *Y/L*, TFPs, & *K/L* jointly emerging as (only) stable patterns through ***symmetry-breaking*** due to ***two-way causality***
  - An explanation for ***Great Divergence***, ***Growth Miracle***

- Most existing studies in 2-country/2-tradeables. In many countries,
  - Does symmetry-breaking split the world into the rich-poor clusters (a polarization)? Or
  - keep splitting into finer clusters until they become more dispersed & fully ranked?
  - What determines the shape of the distribution generated by this mechanism?

  Absent analytical results, the message is unclear. In addition,
  - Comparative statics?
  - Welfare implications?

In this paper,

- An ***analytically solvable*** symmetry-breaking model of trade & inequality with many countries

- **Main Ingredients** of the model
  - ➢A finite number (J) of (ex-ante) identical countries (or regions)
  - ➢A continuum of tradeable consumption goods, $s \in [0,1]$, with Cobb-Douglas preferences (with the uniform expenditure share, wlog)
  - ➢Tradeables produced with Cobb-Douglas tech. with the share of nontradeable intermediate inputs "producer services," $\gamma(s)$, (increasing, wlog; assumed C^1)
  - ➢TFP of the service sector increasing in its size, due to external economies
    (In Ecta, the service sector is monopolistic competitive with variety effect.)

- **Symmetry-Breaking: Two-way causality** between patterns of trade and productivity
  - ➢A country with a large service sector is not only more productive, but also has CA in tradeables which depend more on services.
  - ➢Having CA in those tradeables means a larger market for such services.

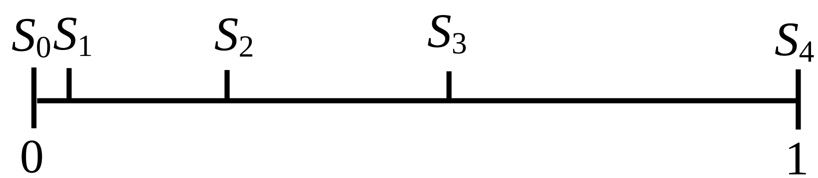
- **What makes the model tractable:** Countries are vastly ***outnumbered*** by tradeables (Countries are much larger than sectors, even if J is arbitrary large)

A Preview of the Main Results

- **Endogenous comparative advantage**: For any finite *J*, countries sort themselves into different tradeable goods in any *stable* equilibrium;

  ➢ A unit interval [0,1] is partitioned into *J* subintervals.

Illustrated for *J* = 4

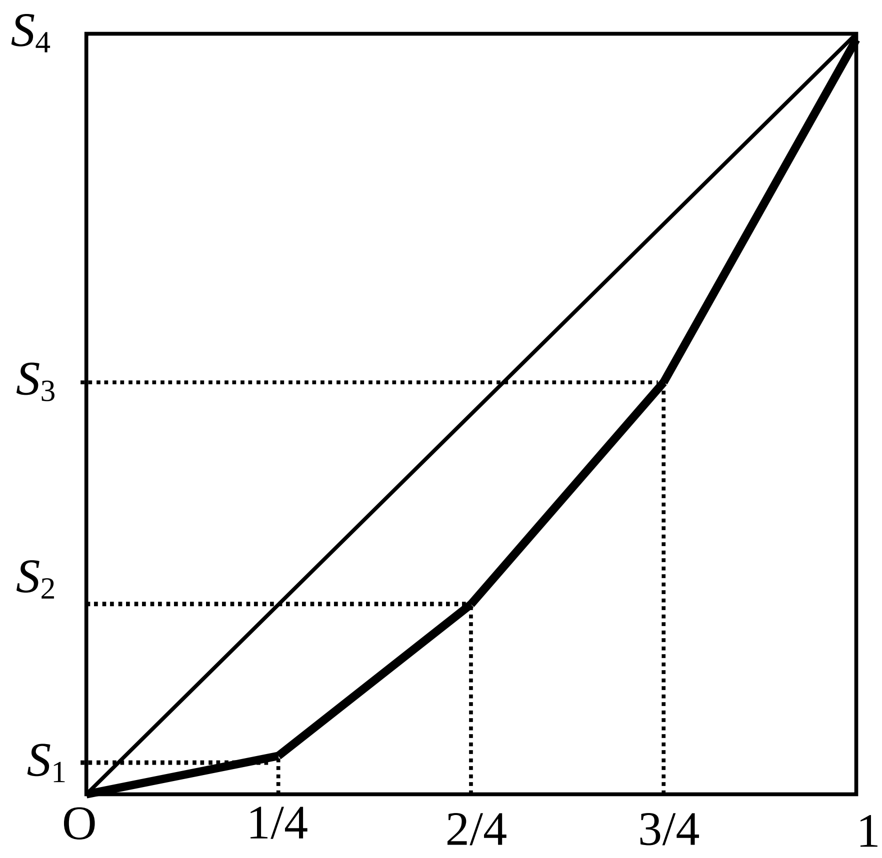


S_j: (Cumulative) share of the *j* poorest countries, characterized by **2nd order difference equation with the 2 terminal conditions**

NB: The subintervals are monotone increasing in length

➢ **Strict ranking** of countries in *Y/L*, TFP, and *K/L*, which are (perfectly) correlated.

Equilibrium Lorenz curve, Φ^J: Illustrated for J = 4



- As $J \to \infty$, the limit Lorenz curve is given by the unique solution of the **2nd order differential equation with the 2 terminal conditions.** Furthermore, analytically solvable.

➢ **Shape of Lorenz Curve:** conditions for
  ✓ Bimodal distribution (Polarization)
  ✓ Power-law distribution

➢ **Comparative Statics:**
  log-super(sub)modularity → Lorenz-dominance

- **Welfare effects of trade:** We can also answer questions like;

➢ When is trade Pareto-improving?
➢ If not Pareto-improving, what fractions of countries would lose from trade?

Organization of this presentation slides:

1. Introduction
2. Basic Model (Fixed Factor Supply; Without Nontradeable Consumption Goods)
   - Single-country (Autarky) equilibrium ($J = 1$)
   - Two-country equilibrium ($J = 2$), not in the paper
   - Multi-country equilibrium ($2 \leq J < \infty$)
   - Limit case ($J \to \infty$)
     - ✓ Polarization
     - ✓ Power-law (truncated Pareto) examples
     - ✓ Comparative Statics; Log-modularity and Lorenz-dominance
3. Welfare Effects of Trade
   - Multi-country equilibrium ($2 \leq J < \infty$)
   - Limit case ($J \to \infty$)
4. Formal Stability Analysis, not in the paper
5. Nontradeable Consumption Goods; Effects of Globalization through Goods Trade
   - Multi-country equilibrium ($2 \leq J < \infty$)
   - Limit case ($J \to \infty$)
6. Variable Factor Supply; Effects of Globalization through Factor Mobility or Skill-Biased Technological Change
   - Multi-country equilibrium ($2 \leq J < \infty$)
   - Limit case ($J \to \infty$)
7. Concluding Remarks, not in the paper

2. Baseline Model: All Factors in Fixed Supply, All Consumer Goods Tradeable

J (inherently) identical countries

Representative Consumers:

- Endowed with V units of the (nontradeable) primary factor of production, which may be a composite of many factors, as $V = F(K, L, \ldots)$.

- Cobb-Douglas preferences over **Tradeable Consumer Goods**, $s \in [0,1]$

$$\log U = \int_0^1 \log(X(s))dB(s) = \int_0^1 \log(X(s))ds,$$

indexed so that B(s) = s, wlog.

Tradeable Consumer Goods Sectors $s \in [0,1]$: *Competitive, CRS*

Unit cost function: $C(s) = \zeta(s)(\omega)^{1-\gamma(s)}(P_N)^{\gamma(s)}$

ω: price of the primary factor of production (Aggregate TFP in equilibrium).
P_N: price of nontradeable producer services
$\gamma(s)$: share of services in sector-s, increasing in $s \in [0,1]$, wlog; assumed to be C^1

Nontradeable Producer Services Sector: *Competitive, External Economies of Scale*

Unit cost function: $P_N = \dfrac{\omega}{A(n)}$,

$A(n)$: Sectoral TFP, increasing in the total input in the sector, n, with

Degree of Scale Economies, $\theta(n) \equiv \dfrac{A'(n)n}{A(n)} > 0$, continuous. ($\theta(n) = \theta > 0$ in Ecta.)

Unit Cost in Sector-s: $C(s) = \zeta(s)(A(n))^{-\gamma(s)}\omega$

High-indexed sectors benefit more from scale economies in the service sector

In stable equilibrium, ω and n will end up being different across countries.

Single-country* (*J* = 1) *or Autarky Case: The economy produces all $s \in [0,1]$.

Let $\Gamma^A \equiv \int_0^1 \gamma(s)ds$.

$$n^A\omega^A = \Gamma^A Y^A$$

$$Y^A = \omega^A V = \omega^A F(K,L,\ldots)$$

$$\rightarrow \quad n^A = V\,\Gamma^A$$

Generally,

The size for the service sector is proportional to the average share of services across all (active) tradeable goods sector.

Two-Country* (*J* = 2) *Case: Home & Foreign (*). **Suppose *n* < *n**.** Then,

- $\dfrac{C(s)}{C^*(s)} = \left(\dfrac{A(n)}{A(n^*)}\right)^{-\gamma(s)}\left(\dfrac{\omega}{\omega^*}\right)$, increasing in *s*.

A country with a higher *n* has comparative advantage in higher-indexed sectors.

- H exports $s \in [0, S)$ & F exports $s \in (S, 1]$, where $\dfrac{C(S)}{C^*(S)} = \left(\dfrac{A(n)}{A(n^*)}\right)^{-\gamma(S)}\left(\dfrac{\omega}{\omega^*}\right) = 1$

Each country must be the least cost producer for a positive measure of tradeables.

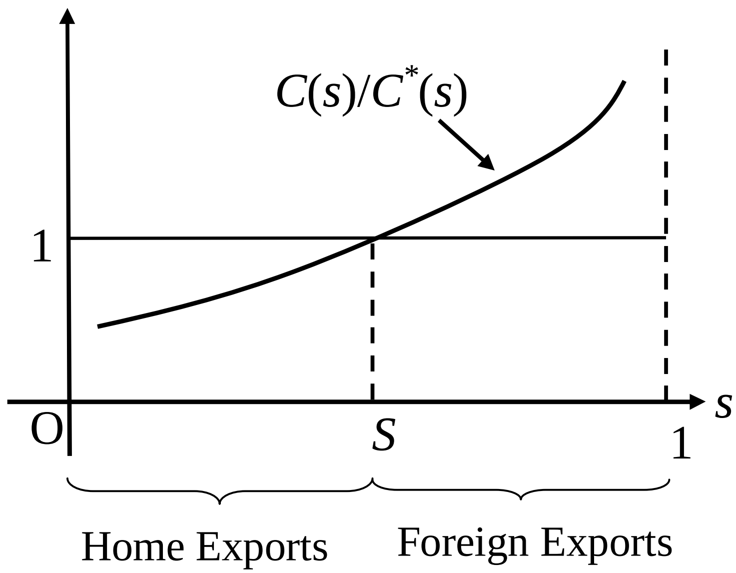
- $\dfrac{\omega}{\omega^*} = \left(\dfrac{A(n)}{A(n^*)}\right)^{\gamma(S)} < 1.$
- $S(Y+Y^*) = Y = \omega V$ & $(1-S)(Y+Y^*) = Y^* = \omega^* V$

A country's share = the world's expenditure share of the consumer goods it produces.

- $n = \left[\dfrac{1}{S}\int_0^S \gamma(s)ds\right]V < n^* = \left[\dfrac{1}{1-S}\int_S^1 \gamma(s)ds\right]V$

The service sector in each country is proportional to the average share of services among its (active) tradeable sectors.

$$\Rightarrow \frac{S}{1-S} = \frac{Y}{Y^*} = \frac{\omega}{\omega^*} = \left(\frac{A(\Gamma^-(S)V)}{A(\Gamma^+(S)V)}\right)^{\gamma(S)} < 1.$$



A Symmetric Pair of Stable Asymmetric Equilibria

- Home produces $s \in [0, S]$ and Foreign produces $s \in [S, 1]$,

$$\frac{Y}{Y^*} = \frac{\omega}{\omega^*} = \frac{S}{1-S} = \left(\frac{A(\Gamma^-(S)V)}{A(\Gamma^+(S)V)}\right)^{\gamma(S)} < 1;$$

- Foreign produces $s \in [0, S]$ and Home produces $s \in [S, 1]$,

$$\frac{Y}{Y^*} = \frac{\omega}{\omega^*} = \frac{1-S}{S} = \left(\frac{A(\Gamma^+(S)V)}{A(\Gamma^-(S)V)}\right)^{\gamma(S)} > 1.$$

Instability of Symmetric Equilibrium: $n = n^*$ (= n^A)

Stable Equilibrium Patterns in the *J*-Country World:

Index the countries such that $\{n_j\}_{j=1}^J$ is monotone increasing. Then,

- $\dfrac{C_j(s)}{C_{j+1}(s)} = \left(\dfrac{A(n_j)}{A(n_{j+1})}\right)^{-\gamma(s)} \left(\dfrac{\omega_j}{\omega_{j+1}}\right)$, is strictly increasing in *s:*

  A country with a higher *n* has comparative advantage in higher-indexed sectors.
- The *j*-th exports $s \in (S_j, S_{j+1})$, where $\{S_j\}_{j=1}^J$ is monotone increasing, with $S_0 = 0$, $S_J = 1$

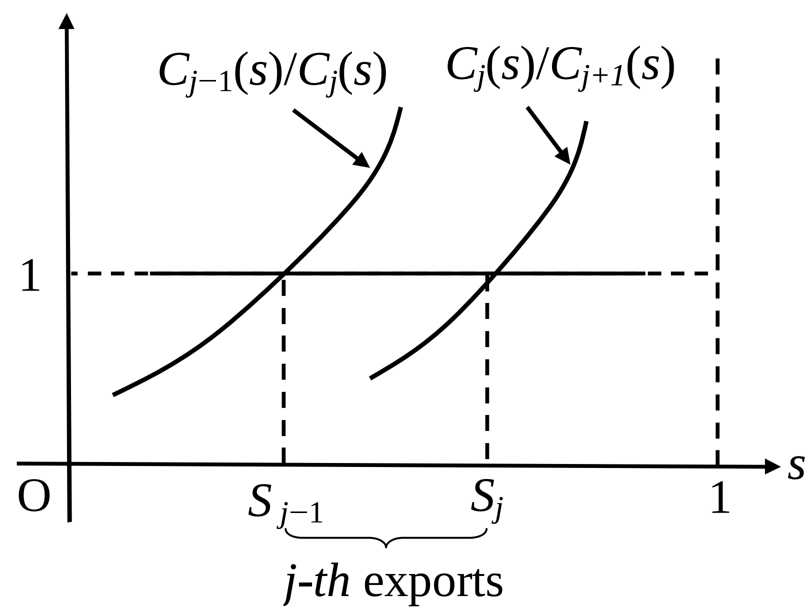
  and $\dfrac{C_j(S_j)}{C_{j+1}(S_j)} = \left(\dfrac{A(n_j)}{A(n_{j+1})}\right)^{-\gamma(S_j)} \left(\dfrac{\omega_j}{\omega_{j+1}}\right) = 1.$ → $\{\omega_j\}_{j=1}^J$ is monotone increasing.
- $Y_j = \omega_j F(K, L, \ldots) = (S_j - S_{j-1})Y^W$

A country's share = world's expenditure share of the consumer goods it produces.

- $n_j = V\Gamma_j$, where $\Gamma_j \equiv \dfrac{1}{S_j - S_{j-1}} \displaystyle\int_{S_{j-1}}^{S_j} \gamma(s)ds$

The service sector in each country is proportional to the average expenditure share on such services among its (active) tradeable sectors.

→ $\{n_j\}_{j=1}^J$ monotone increasing, as assumed.



This can be summarized as:

Proposition 1 (*J*-country case):

$\{S_j\}_{j=0}^{J}$ solves the nonlinear 2nd-order difference equation with the 2 terminal conditions:

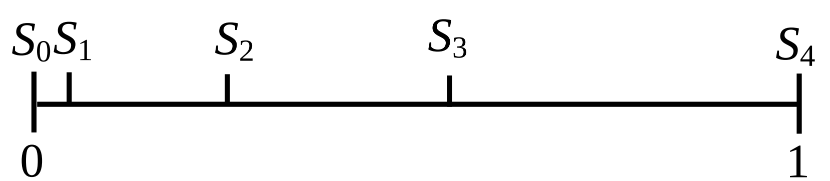
$$\frac{S_{j+1}-S_j}{S_j-S_{j-1}}=\left(\frac{A(V\Gamma(S_j,S_{j+1}))}{A(V\Gamma(S_{j-1},S_j))}\right)^{\gamma(S_j)}>1 \text{ with } S_0=0 \ \& \ S_J=1,$$

where $$\Gamma(S_{j-1},S_j)\equiv\frac{1}{S_j-S_{j-1}}\int_{S_{j-1}}^{S_j}\gamma(s)ds.$$

In Ecta, $A(n)\propto(n)^{\theta}$ →

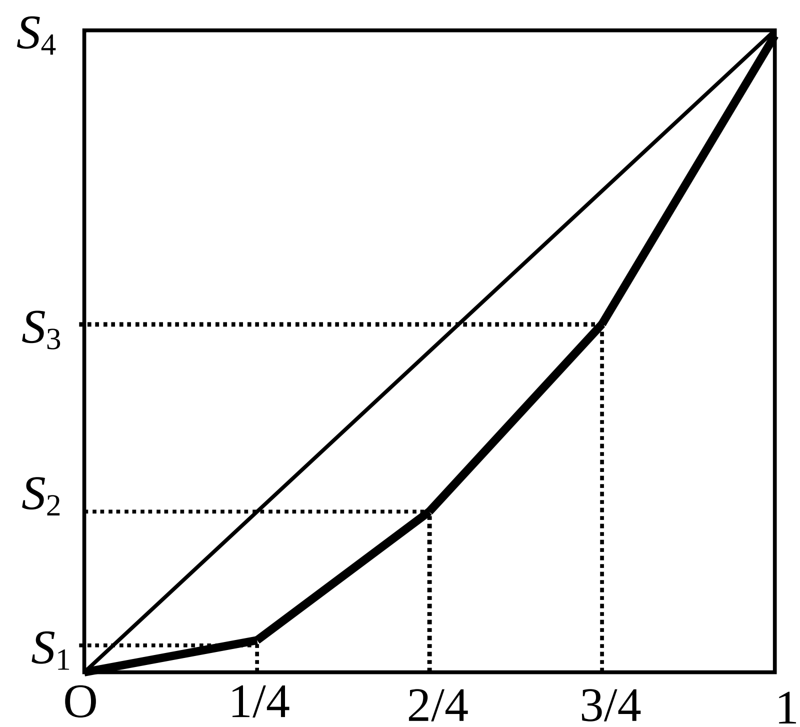
$$\frac{S_{j+1}-S_j}{S_j-S_{j-1}}=\left(\frac{\Gamma(S_j,S_{j+1}))}{\Gamma(S_{j-1},S_j)}\right)^{\theta\gamma(S_j)}>1$$, scale-free, independent of *V*.

Illustrated for *J* = 4



The Lorenz curve, $\Phi^J : [0,1] \to [0,1]$, is the piece-wise linear function, $\Phi^J(j/J) = S_j$.

- Φ^J is strictly increasing & convex;
- $\Phi^J(0) = 0$ & $\Phi^J(1) = 1$.



But, it is not analytically solvable.

- Uniqueness?
- Comparative statics?
- Welfare evaluations?

These problems disappear by $J \to \infty$.

Calculating the limit Lorenz Curve, Φ, which turns out to be C^2

$$\frac{S_{j+1}-S_j}{S_j - S_{j-1}} = \left(\frac{A(V\Gamma(S_j,S_{j+1}))}{A(V\Gamma(S_{j-1},S_j))}\right)^{\gamma(S_j)} \quad \text{with } \Gamma(S_{j-1},S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s)ds$$

By setting $x = j/J$ and $\Delta x = 1/J$,

$$S_{j+1} - S_j = \Phi(x+\Delta x) - \Phi(x) = \Phi'(x)\Delta x + \Phi''(x)\tfrac{|\Delta x|^2}{2} + o\left(|\Delta x|^2\right),$$

$$S_j - S_{j-1} = \Phi(x) - \Phi(x-\Delta x) = \Phi'(x)\Delta x - \Phi''(x)\tfrac{|\Delta x|^2}{2} + o\left(|\Delta x|^2\right),$$

from which

$$\text{LHS} = \frac{S_{j+1}-S_j}{S_j - S_{j-1}} = 1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o(|\Delta x|).$$

Likewise,

$$\Gamma(S_j,S_{j+1}) = \frac{\int_{\Phi(x)}^{\Phi(x+\Delta x)} \gamma(s)ds}{\Phi(x+\Delta x) - \Phi(x)} = \gamma(\Phi(x)) + \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

$$\Gamma(S_{j-1},S_j) = \frac{\int_{\Phi(x-\Delta x)}^{\Phi(x)} \gamma(s)ds}{\Phi(x) - \Phi(x-\Delta x)} = \gamma(\Phi(x)) - \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

so that

$$A\left(V\Gamma(S_j, S_{j+1})\right) = A\left(V\gamma(\Phi(x))\right) + \frac{V}{2}A'\left(V\gamma(\Phi(x))\right)\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

$$A\left(V\Gamma(S_{j-1}, S_j)\right) = A\left(V\gamma(\Phi(x))\right) - \frac{V}{2}A'\left(V\gamma(\Phi(x))\right)\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|),$$

from which

$$\text{RHS} = \left(\frac{A(V\Gamma(S_j, S_{j+1}))}{A(V\Gamma(S_{j-1}, S_j))}\right)^{\gamma(S_j)} = \left(1 + \frac{VA'(V\gamma(\Phi(x)))}{A(V\gamma(\Phi(x)))}\gamma'(\Phi(x))\Phi'(x)\Delta x + o\left(|\Delta x|\right)\right)^{\gamma(\Phi(x))}$$

$$= 1 + \theta(V\gamma(\Phi(x)))\gamma'(\Phi(x))\Phi'(x)\Delta x + o\left(|\Delta x|\right)$$

where $\theta(n) \equiv \dfrac{A'(n)n}{A(n)} > 0$.

Combining these yields

$$1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o\left(|\Delta x|\right) = 1 + \theta(V\gamma(\Phi(x)))\gamma'(\Phi(x))\Phi'(x)\Delta x + o\left(|\Delta x|\right).$$

Hence, as $J \to \infty$, $\Delta x = 1/J \to 0$,

$$\frac{\Phi''(x)}{\Phi'(x)} = \theta(V\gamma(\Phi(x)))\gamma'(\Phi(x))\Phi'(x)$$

By integrating once,

$$\log(\Phi'(x)) - \frac{\Theta(V\gamma(\Phi(x)))}{V} = c_0, \qquad \text{where } \Theta(n) \equiv \int_0^n \theta(u)du .$$

This can be rewritten as

$$\exp\left(\frac{-\Theta(V\gamma(\Phi(x)))}{V}\right)\Phi'(x) = e^{c_0}$$

By integrating once again,

$$\int_0^{\Phi(x)} \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)ds = c_1 + e^{c_0}x .$$

From $\Phi(0) = 0$ & $\Phi(1) = 1$,

$$\int_0^{\Phi(x)} \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)ds = \left[\int_0^1 \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)ds\right]x$$

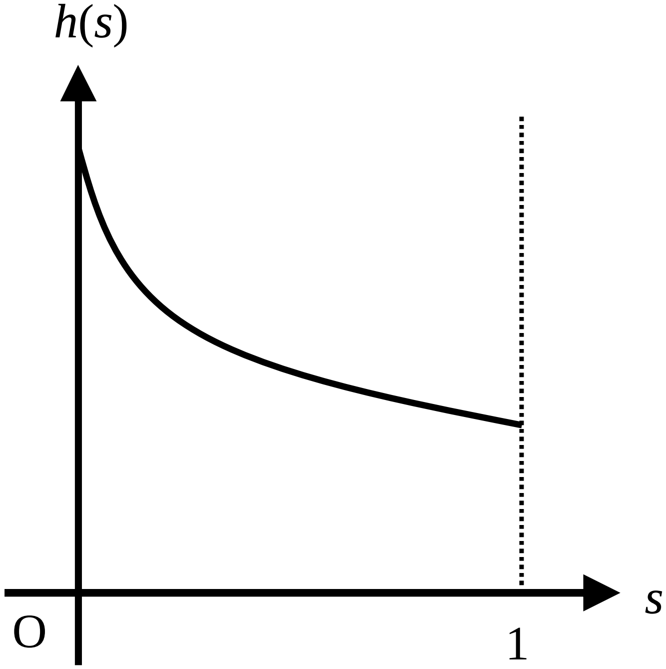
$$\Leftrightarrow \quad x = H(\Phi(x)) \equiv \int_0^{\Phi(x)} h(s)ds, \text{ where } h(s) \equiv \frac{\hat{h}(s)}{\int_0^1 \hat{h}(u)du} \quad \text{with } \hat{h}(s) \equiv \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)$$

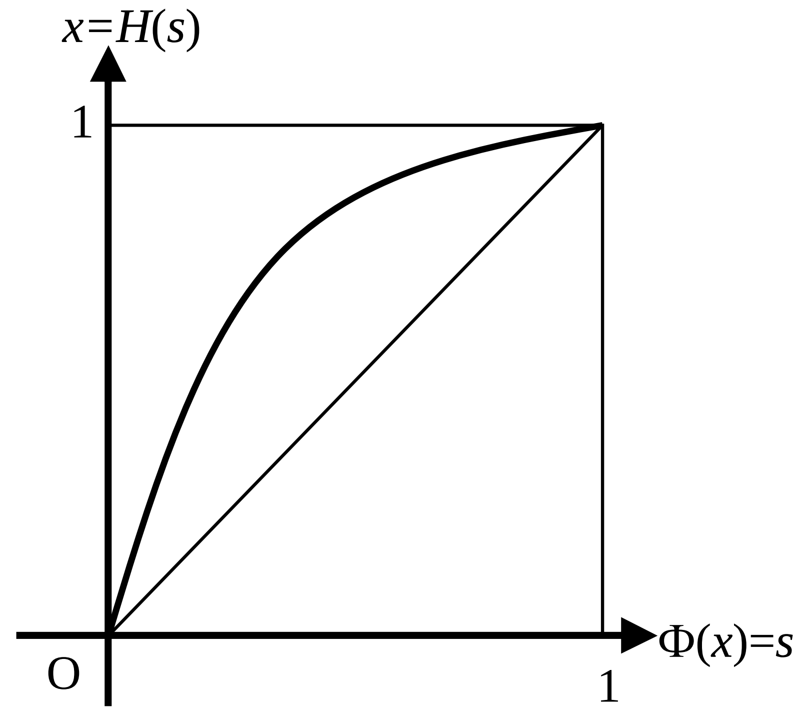
.

In Ecta, $A(n) \propto (n)^{\theta}$, $\theta(n) = \theta$ → $\Theta(n) = \theta n$, → $\hat{h}(s) = \exp(-\theta\gamma(s))$, scale-free.

Proposition 2 (Limit Case: $J \rightarrow \infty$) The limit equilibrium Lorenz curve, Φ, is given by

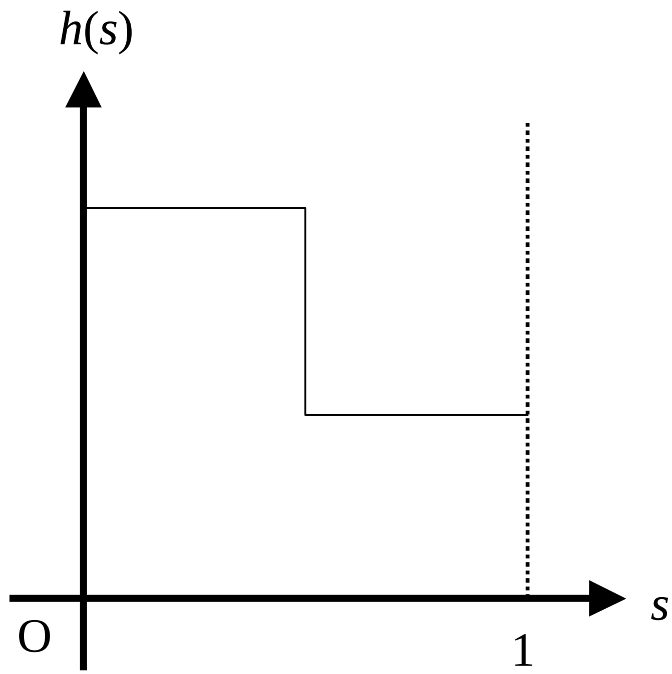
$$x = H(\Phi(x)) \equiv \int_0^{\Phi(x)} h(s)ds, \text{ where } h(s) \equiv \frac{\hat{h}(s)}{\int_0^1 \hat{h}(u)du} \quad \text{with } \hat{h}(s) \equiv \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)$$

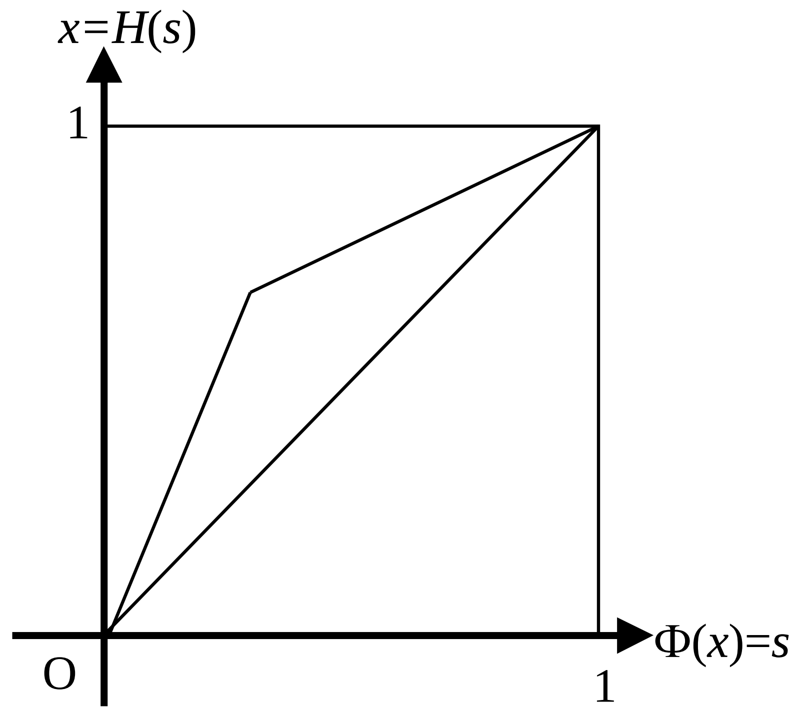




NB: Lorenz Curve also maps a set of countries into a set of goods they produced

Question: When does this mechanism lead to *a polarization*?





Answer: When $\Theta(V\gamma(s))$ is *approximately* a two-step function. That is, either when

- $\gamma(\bullet)$ is approximately a two-step (e.g., effectively there are only two tradeables)

Note: This is different from assuming that there are only two tradeable goods. The uniqueness is lost when you do that.

- $\Theta(\bullet)$ is approximately a two-step (e.g., the single threshold externalities).

Power-Law (Truncated Pareto) Examples (with World GDP normalized on one):

	Example 1: $\gamma(s)=s$	Example 2: $\gamma(s)=\log\left[1+(e^{\theta}-1)s\right]^{\frac{1}{\theta}}$	Example 3: $\gamma(s)=\log\left[1+(e^{\lambda}-1)s\right]^{\frac{1}{\lambda}}$ $(\lambda\neq 0;\neq\theta)$
Inverse Lorenz Curve: $x=H(s)$	$\dfrac{1-e^{-\theta s}}{1-e^{-\theta}}$	$\log\left[1+(e^{\theta}-1)s\right]^{\frac{1}{\theta}}$	$\dfrac{\left[1+(e^{\lambda}-1)s\right]^{1-\frac{\theta}{\lambda}}-1}{e^{\lambda-\theta}-1}$
Lorenz Curve: $s=\Phi(x)$	$\log\left[1-(1-e^{-\theta})x\right]^{-\frac{1}{\theta}}$	$\dfrac{e^{\theta x}-1}{e^{\theta}-1}$	$\dfrac{\left[1+(e^{\lambda-\theta}-1)x\right]^{\frac{\lambda}{\lambda-\theta}}-1}{e^{\lambda}-1}$
Cdf: $x=\Psi(y)$ $=(\Phi')^{-1}(y)$	$\dfrac{1}{1-e^{-\theta}}-\dfrac{1}{\theta y}$	$\dfrac{1}{\theta}\log\left(\dfrac{e^{\theta}-1}{\theta}y\right)$	$\dfrac{\left(\dfrac{y}{y_{Min}}\right)^{\frac{\lambda}{\theta}-1}-1}{e^{\lambda-\theta}-1}=1-\dfrac{1-\left(\dfrac{y}{y_{Max}}\right)^{\frac{\lambda}{\theta}-1}}{1-e^{\theta-\lambda}}$
Pdf: $\psi(y)=\Psi'(y)$	$\dfrac{1}{\theta y^{2}}$	$\dfrac{1}{\theta y}$	$\left[\dfrac{(\lambda/\theta)-1}{(y_{Max})^{(\lambda/\theta)-1}-(y_{Min})^{(\lambda/\theta)-1}}\right](y)^{\frac{\lambda}{\theta}-2}$
Support: $[y_{Min},y_{Max}]$	$\dfrac{1-e^{-\theta}}{\theta}\leq y\leq\dfrac{e^{\theta}-1}{\theta}$	$\dfrac{\theta}{e^{\theta}-1}\leq y\leq\dfrac{\theta e^{\theta}}{e^{\theta}-1}$	$\left(\dfrac{\lambda}{e^{\lambda}-1}\right)\left(\dfrac{e^{\lambda-\theta}-1}{\lambda-\theta}\right)\leq y\leq\left(\dfrac{\lambda}{e^{\lambda}-1}\right)\left(\dfrac{e^{\lambda-\theta}-1}{\lambda-\theta}\right)e^{\theta}$

A lower λ (more concentrated use of services in narrower sectors) makes the pdf drop faster.

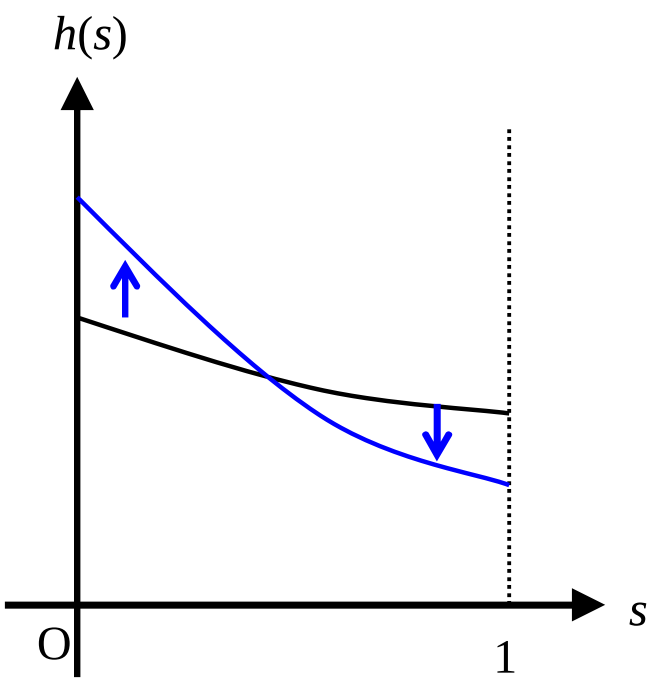
Log-modularity and Lorenz Dominance(not in the paper)

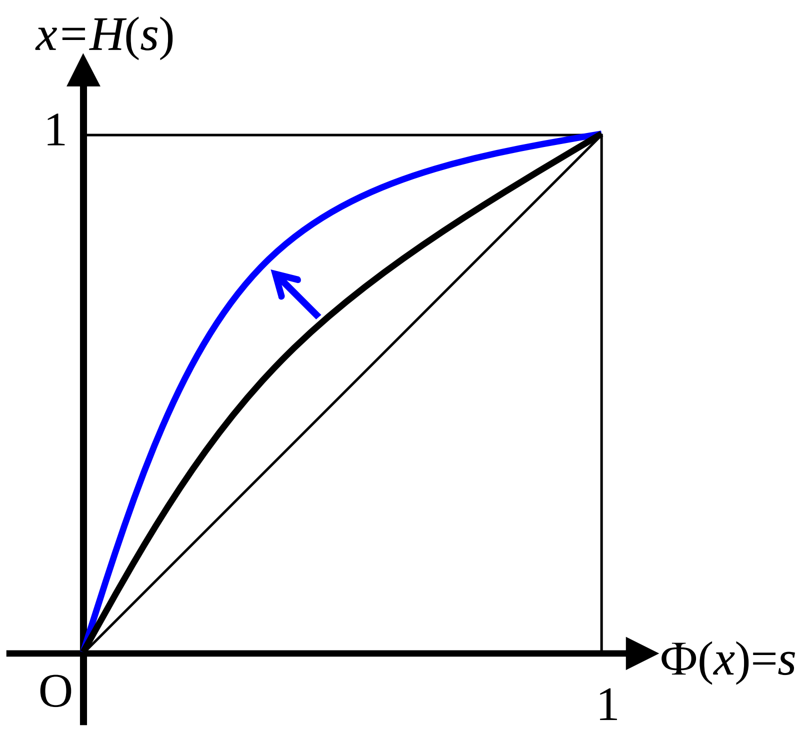
Lemma: For a positive value function, $\hat{h}(\bullet;\sigma)$: [0,1] → R_+, with a parameter σ, define

$H(\bullet;\sigma)$: [0,1] → [0,1], by $H(s;\sigma) \equiv \int_0^s h(u;\sigma)du = \dfrac{\int_0^s \hat{h}(u;\sigma)du}{\int_0^1 \hat{h}(u;\sigma)du}$. Then,

$$\frac{\partial H}{\partial \sigma} > (<) 0 \text{ if } \hat{h}(s;\sigma) \text{ is } \textit{log-sub(super)modular} \text{ in } \sigma \text{ and } s.$$

An Illustration: Log-Submodular Case





Proof:

From $\frac{1}{H}\frac{\partial H}{\partial \sigma} = \psi(s) - \psi(1)$, where $\psi(s) \equiv \dfrac{\int_0^s \hat{h}_\sigma(u;\sigma)du}{\int_0^s \hat{h}(u;\sigma)du}$.

$$\psi'(s) = \frac{\hat{h}_\sigma(s;\sigma)\int_0^s \hat{h}(u;\sigma)du - \hat{h}(s;\sigma)\int_0^s \hat{h}_\sigma(u;\sigma)du}{\left[\int_0^s \hat{h}(u;\sigma)du\right]^2} = \frac{\int_0^s \left[\hat{h}_\sigma(s;\sigma)\hat{h}(u;\sigma) - \hat{h}(s;\sigma)\hat{h}_\sigma(u;\sigma)\right]du}{\left[\int_0^s \hat{h}(u;\sigma)du\right]^2}$$

$$= \frac{\int_0^s \left[\frac{\partial \ln \hat{h}(s;\sigma)}{\partial \sigma} - \frac{\partial \ln \hat{h}(u;\sigma)}{\partial \sigma}\right]\hat{h}(s;\sigma)\hat{h}(u;\sigma)du}{\left[\int_0^s \hat{h}(u;\sigma)du\right]^2} = \frac{\int_0^s \left[\int_u^s \frac{\partial^2 \ln \hat{h}(v;\sigma)}{\partial \sigma \partial v}dv\right]\hat{h}(s;\sigma)\hat{h}(u;\sigma)du}{\left[\int_0^s \hat{h}(u;\sigma)du\right]^2}$$

Hence,

$\frac{\partial^2 \ln \hat{h}}{\partial \sigma \partial s} < 0$ implies $\psi'(s) < 0$ and $\frac{1}{H}\frac{\partial H}{\partial \sigma} = \psi(s) - \psi(1) = -\int_s^1 \psi'(u)du > 0$.

$\frac{\partial^2 \ln \hat{h}}{\partial \sigma \partial s} > 0$ implies $\psi'(s) > 0$ and $\frac{1}{H}\frac{\partial H}{\partial \sigma} = \psi(s) - \psi(1) = -\int_s^1 \psi'(u)du < 0$. **Q.E.D.**

Since $\frac{\partial \ln \hat{h}(s;V)}{\partial s} = -\theta\big(\gamma(s)V\big)\gamma'(s)$,

Effect of a higher *V*:

- For $\theta'(n) > 0$, $\ln \hat{h}(s;V)$ is *submodular* in *V* & *s*. Thus, a higher *V* → more inequality.
- For $\theta'(n) < 0$, $\ln \hat{h}(s;V)$ is *supermodular* in *V* & *s*. Thus, a higher *V* → less inequality.

Effect of a higher *θ*:

In Ecta,
$\theta(n) = \theta > 0$, $\ln \hat{h}(s;\theta) = -\theta\gamma(s)$ is *submodular* in *θ* & *s*. Thus, a higher *θ* → more inequality.

NB: This also works for any shift parameter, σ, such $\theta_\sigma(n;\sigma) > 0$.

3. Welfare Effects of Trade

$$\log\left(U^A\right) = \log\left(\omega^A V\right) - \int_0^1 \log\left(P^A(s)\right)ds.$$

$$\log\left(U_j\right) = \log\left(\omega_j V\right) - \int_0^1 \log\left(P(s)\right)ds,$$

$$\frac{P(s)}{P^A(s)} = \left(\frac{\omega_k}{\omega^A}\right)\left(\frac{A(n_k)}{A(n^A)}\right)^{-\gamma(s)} = \left(\frac{\omega_k}{\omega^A}\right)\left(\frac{A(V\Gamma_k)}{A(V\Gamma^A)}\right)^{-\gamma(s)}$$ for $s \in (S_{k-1}, S_k)$ for $k = 1, 2, \ldots, J$.

Combining these yields

$$\log\left(\frac{U_j}{U^A}\right) = \log\left(\frac{\omega_j}{\omega^A}\right) - \sum_{k=1}^{J}\left[\int_{S_{k-1}}^{S_k} \log\left(\frac{\omega_k}{\omega^A}\right)ds - \int_{S_{k-1}}^{S_k} \gamma(s)\log\left(\frac{A(V\Gamma_k)}{A(V\Gamma^A)}\right)ds\right],$$

which can be rewritten as:

Proposition 3 (*J*-country case): The welfare of the j-th poorest country is

$$\log\left(\frac{U_j}{U^A}\right) = \sum_{k=1}^{J}\log\left(\frac{\omega_j}{\omega_k}\right)\left(S_k - S_{k-1}\right) + \sum_{k=1}^{J}\Gamma_k \log\left(\frac{A(V\Gamma_k)}{A(V\Gamma^A)}\right)\left(S_k - S_{k-1}\right)$$

- 1st term: productivity dispersion effect, negative for some countries.
- 2nd term; gains from trade (conditional on productivity dispersion), always positive.

By setting $x^* = j/J$ and $x = k/J$ and noting that, as $J \to \infty$,

$$\omega_j / \omega_k \to \Phi'(x^*)/\Phi'(x) \text{ and } S_k - S_{k-1} \to \Phi'(x)dx,$$

$$\log\left(\frac{U(x^*)}{U^A}\right) = \int_0^1 \log\left(\frac{\Phi'(x^*)}{\Phi'(x)}\right)\Phi'(x)dx + \int_0^1 \gamma(\Phi(x))\log\left(\frac{A(V\gamma(\Phi(x)))}{A(V\Gamma_A)}\right)\Phi'(x)dx.$$

From $\log(\Phi'(x)) - \dfrac{\Theta(V\gamma(\Phi(x)))}{V} = c_0$,

$$\log\left(\frac{U(x^*)}{U^A}\right) = \int_0^1 \left(\frac{\Theta(V\gamma(\Phi(x^*))}{V} - \frac{\Theta(V\gamma(\Phi(x))}{V}\right)d\Phi + \int_0^1 \gamma(\Phi(x))\log\left(\frac{A(V\gamma(\Phi(x)))}{A(V\Gamma_A)}\right)d\Phi,$$

Or

Proposition 4 (Limit case, $J \to \infty$): The welfare of the country at $100x^*\%$ is given by

$$\log\left(\frac{U(x^*)}{U^A}\right) = \frac{\Theta(V\gamma(s^*))}{V} - \int_0^1 \left(\frac{\Theta(V\gamma(s))}{V}\right)ds + \int_0^1 \gamma(s)\log\left(\frac{A(V\gamma(s))}{A(V\Gamma_A)}\right)ds$$

where $s^* = \Phi(x^*)$ or $x^* = \Phi^{-1}(s^*)$.

- 1st two terms; productivity dispersion effect, negative for some countries.
- 3rd term; gains from trade (conditional on productivity dispersion), always positive.

Corollary 1: All countries gain from trade iff

$$\int_0^1 \left(\frac{\Theta(V\gamma(s))}{V} \right) ds - \frac{\Theta(V\gamma(0))}{V} < \int_0^1 \gamma(s) \log\left(\frac{A(V\gamma(s))}{A(V\Gamma_A)} \right) ds$$

In Ecta, $A(n) \propto (n)^{\theta}$, $\theta(n) = \theta$, this can be rewritten as:

$$1 - \frac{\gamma(0)}{\Gamma^A} \leq \int_0^1 \left(\frac{\gamma(s)}{\Gamma^A} \right) \log\left(\frac{\gamma(s)}{\Gamma^A} \right) ds$$ = diversity (**Theil index/entropy**) of γ.

Corollary 2: Suppose the condition of Corollary 1 fails. Define $s_c \in (0,1)$ by

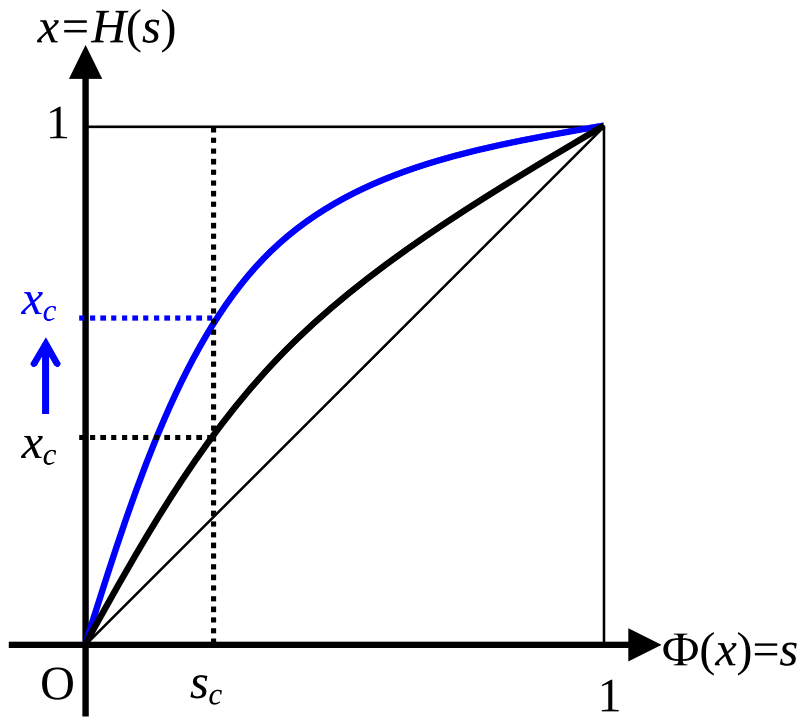
$$\frac{\Theta(V\gamma(s_c))}{V} \equiv \int_0^1 \left(\frac{\Theta(V\gamma(s))}{V} \right) ds - \int_0^1 \gamma(s) \log\left(\frac{A(V\gamma(s))}{A(V\Gamma_A)} \right) ds.$$

a): All countries producing $s \in [0, s_c)$ lose from trade.

b): Consider a shift parameter, σ > 0, such that $A(n;\sigma) = [A(n)]^{\sigma}$. Then, s_c is independent of σ , and the fraction of the countries that lose, $x_c = H(s_c;\sigma)$, is increasing in σ with $\lim_{\sigma\to 0} x_c = s_c$ and $\lim_{\sigma\to\infty} x_c = 1$.

In Ecta, $A(n) \propto (n)^{\theta}$, hence $\theta(n) = \theta$ is used as the shift parameter, σ.

Corollary 2: A Graphic Illustration



4. Formal Stability Analysis in a Dynamic Model with Learning-By-Doing Externalities

Competitive Nontradeable Producer Services Sector: $p_N = \omega/A(Q)$

$A(Q)$: Sectoral TFP, increasing in Q, *the cumulative experience as defined later.*

Stable Equilibrium Patterns in the *J*-Country World:

Index the countries so $\{Q_j\}_{j=1}^{J}$ is monotone increasing. Then,

- The unit interval is partitioned into J-subintervals: the j-th exports $s \in (S_j, S_{j+1})$, where $\{S_j\}_{j=1}^{J}$ is given by $S_0 = 0$, $S_J = 1$
- $\dfrac{S_{j+1}-S_j}{S_j - S_{j-1}} = \left(\dfrac{A(Q_{j+1})}{A(Q_j)}\right)^{\gamma(S_j)} > 1$ with $S_0 = 0$ & $S_J = 1$.
- $n_j = \Gamma(S_{j-1}, S_j)V$.

Learning-By-Doing Externalities: Country-specific experience is measured by the discounted labor input in the past:

$$Q_j(t) = \delta\int_{-\infty}^{t} n_j(v)\exp[\delta(v-t)]dv \quad \rightarrow \quad \dot{Q}_j(t) = \delta\left(n_j(t) - Q_j(t)\right)$$

Dynamics: Given $\{Q_j(t)\}_{j=1}^J$, monotone increasing,

- $\dfrac{S_{j+1}(t)-S_j(t)}{S_j(t)-S_{j-1}(t)} = \left(\dfrac{A(Q_{j+1}(t))}{A(Q_j(t))}\right)^{\gamma(S_j)} > 1$ with $S_0(t)=0$ & $S_J(t)=1$.
- $\dot{Q}_j(t) = \delta\big(n_j(t)-Q_j(t)\big)$ with $n_j(t) = \Gamma(S_{j-1}(t), S_j(t))V$

Steady State: monotone increasing $\{S_j^*\}_{j=1}^J$, so that

$$\frac{S_{j+1}^* - S_j^*}{S_j^* - S_{j-1}^*} = \left(\frac{A(\Gamma(S_j^*, S_{j+1}^*)V)}{A(\Gamma(S_{j-1}^*, S_j^*)V)}\right)^{\gamma(S_j)} > 1$$

With $A(Q) \propto (Q)^\theta$, $\dfrac{S_{j+1}^* - S_j^*}{S_j^* - S_{j-1}^*} = \left(\dfrac{\Gamma(S_j^*, S_{j+1}^*)}{\Gamma(S_{j-1}^*, S_j^*)}\right)^{\theta\gamma(S_j)} > 1.$

5.Nontradeable Consumption Goods: Effects of Globalization through Goods Trade

$$\log U = \tau\int_0^1 \log(X_T(s))ds + (1-\tau)\int_0^1 \log(X_N(s))ds$$

τ; the fraction of the consumption goods that are tradeable.

Assume the same distribution of γ among the tradeables and the nontradeables.

$$\omega_j V = (1-\tau)\omega_j V + \tau(S_j - S_{j-1})Y^W \rightarrow \omega_j V = (S_j - S_{j-1})Y^W$$

$$\omega_j n_j = \Gamma^A(1-\tau)\omega_j V + \Gamma_j \tau(S_j - S_{j-1})Y^W \rightarrow n_j = \left(\tau\Gamma_j + (1-\tau)\Gamma^A\right)V.$$

Thus,

Proposition 5 (the *J*-country case):

Let S_j be the cumulative share of the J poorest countries. Then, $\{S_j\}_{j=0}^J$ solves:

$$\frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{A\left(V\tau\Gamma(S_j, S_{j+1}) + V(1-\tau)\Gamma^A\right)}{A\left(V\tau\Gamma(S_{j-1}, S_j) + V(1-\tau)\Gamma^A\right)}\right)^{\gamma(S_j)} > 1 \text{ with } S_0 = 0 \ \& \ S_J = 1,$$

where $\Gamma(S_{j-1}, S_j) \equiv \dfrac{1}{S_j - S_{j-1}}\displaystyle\int_{S_{j-1}}^{S_j}\gamma(s)ds$.

As before,

$$\text{LHS} = \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = 1 + \frac{\Phi''(x)}{\Phi'(x)} \Delta x + o(|\Delta x|).$$

Likewise,

$$\Gamma(S_j, S_{j+1}) = \frac{\int_{\Phi(x)}^{\Phi(x+\Delta x)} \gamma(s) ds}{\Phi(x+\Delta x) - \Phi(x)} = \gamma(\Phi(x)) + \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

$$\Gamma(S_{j-1}, S_j) = \frac{\int_{\Phi(x-\Delta x)}^{\Phi(x)} \gamma(s) ds}{\Phi(x) - \Phi(x-\Delta x)} = \gamma(\Phi(x)) - \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

so that

$$A\left(V\tau\Gamma_j + V(1-\tau)\Gamma^A\right) = $$
$$A\left(V\tau\gamma(\Phi(x)) + V(1-\tau)\Gamma^A\right) + \frac{V\tau}{2} A'\left(V\tau\gamma(\Phi(x)) + V(1-\tau)\Gamma^A\right)\frac{d(\gamma(\Phi(x)))}{dx}\Delta x + o(|\Delta x|)$$

$$A\left(V\tau\Gamma_{j-1} + V(1-\tau)\Gamma^A\right) = $$
$$A\left(V\tau\gamma(\Phi(x)) + V(1-\tau)\Gamma^A\right) - \frac{V\tau}{2} A'\left(V\tau\gamma(\Phi(x)) + V(1-\tau)\Gamma^A\right)\frac{d(\gamma(\Phi(x)))}{dx}\Delta x + o(|\Delta x|)$$

from which

$$\text{RHS} = 1+\frac{V\tau\gamma(\Phi(x))A'\left(V\tau\gamma(\Phi(x))+V(1-\tau)\Gamma^{A}\right)}{A\left(V\tau\gamma(\Phi(x))+V(1-\tau)\Gamma^{A}\right)}\frac{d(\gamma(\Phi(x)))}{dx}\Delta x+o(|\Delta x|)$$

$$= 1+\frac{\theta\left(V\tau\gamma(\Phi(x))+V(1-\tau)\Gamma^{A}\right)}{1+(1-\tau)\Gamma^{A}/\tau\gamma(\Phi(x))}\frac{d(\gamma(\Phi(x)))}{dx}\Delta x+o(|\Delta x|)$$

Combining these yields

$$1+\frac{\Phi''(x)}{\Phi'(x)}\Delta x+o\left(|\Delta x|\right) = 1+\frac{\theta\left(V\tau\gamma(\Phi(x))+V(1-\tau)\Gamma^{A}\right)}{1+(1-\tau)\Gamma^{A}/\tau\gamma(\Phi(x))}\frac{d(\gamma(\Phi(x)))}{dx}\Delta x+o(|\Delta x|)$$

Hence, as $J\to\infty$, $\Delta x = 1/J \to 0$,

$$\frac{\Phi''(x)}{\Phi'(x)} = \frac{\theta\left(V\tau\gamma(\Phi(x))+V(1-\tau)\Gamma^{A}\right)}{1+(1-\tau)\Gamma^{A}/\tau\gamma(\Phi(x))}\frac{d(\gamma(\Phi(x)))}{dx}$$

Integrating once,

$$\log\left(\Phi'(x)\right)-\int_{0}^{\gamma(\Phi(x))}\frac{\theta\left(V\tau v+V(1-\tau)\Gamma^{A})\right)}{1+(1-\tau)\Gamma^{A}/\tau v}dv = c_0,$$

which can be rewritten as:

$$\exp\left[-\int_{0}^{\gamma(\Phi(x))}\frac{\theta\left(V\tau v+V(1-\tau)\Gamma^{A})\right)}{1+(1-\tau)\Gamma^{A}/\tau v}dv\right]\Phi'(x) = e^{c_0}\cdot$$

Integrating once more,

$$\int_{0}^{\Phi'x)}\exp\left[-\int_{0}^{\gamma(s)}\frac{\theta\left(V\tau+V(1-\tau)\Gamma^{A})\right)}{1+(1-\tau)\Gamma^{A}/\tau v}dv\right]ds = e^{c_0}x+c_1$$

From $\Phi(0)=0$ & $\Phi(1)=1$,

$$\int_0^{\Phi'x)} \exp\left[-\int_0^{\gamma(s)} \frac{\theta\left(V\tau v + V(1-\tau)\Gamma^A)\right)}{1+(1-\tau)\Gamma^A/\tau v} dv\right] ds = \left[\int_0^1 \exp\left[-\int_0^{\gamma(s)} \frac{\theta\left(V\tau v + V(1-\tau)\Gamma^A)\right)}{1+(1-\tau)\Gamma^A/\tau v} dv\right] ds\right] x$$

Proposition 6 (Limit Case, $J \to \infty$):
The limit equilibrium Lorenz curve, Φ, is given by

$$x = H\left(\Phi(x)\right) \equiv \int_0^{\Phi(x)} h(s)ds,$$

where $h(s) \equiv \dfrac{\hat{h}(s)}{\int_0^1 \hat{h}(u)du}$ with $\hat{h}(s) \equiv \exp\left[-\int_0^{\gamma(s)} \dfrac{\theta\left(V\tau v + V(1-\tau)\Gamma^A)\right)}{1+(1-\tau)\Gamma^A/\tau v} dv\right]$

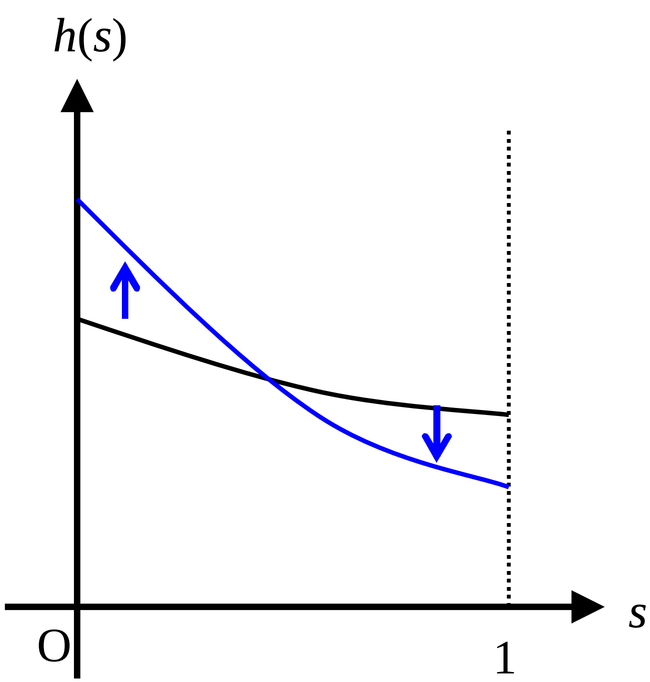
- $\lim_{\tau \to 1} \hat{h}(s) = \exp\left[-\dfrac{\Theta(V\gamma(s))}{V}\right]$; $\lim_{\tau \to 0} \hat{h}(s) = 1.$
- $\ln \hat{h}(s) = -\int_0^{\gamma(s)} \dfrac{\theta\left(V\tau v + V(1-\tau)\Gamma^A)\right)}{1+(1-\tau)\Gamma^A/\tau v} dv$; $\dfrac{\partial \ln \hat{h}(s)}{\partial s} = -\dfrac{\theta\left(V\tau\gamma(s) + V(1-\tau)\Gamma^A)\right)}{1+(1-\tau)\Gamma^A/\tau\gamma(s)} \gamma'(s)$
- $\dfrac{\partial^2 \ln \hat{h}(s)}{\partial V \partial s} < 0$ if $\theta'(n) > 0$, a higher V → more inequality.
- $\dfrac{\partial^2 \ln \hat{h}(s)}{\partial V \partial s} > 0$ if $\theta'(n) < 0$, a higher V → less inequality.
- $\dfrac{\partial^2 \ln \hat{h}(s)}{\partial V \partial s} = 0$ and $\dfrac{\partial^2 \ln \hat{h}(s)}{\partial \theta \partial s} < 0$ for $\theta(n) = \theta$,

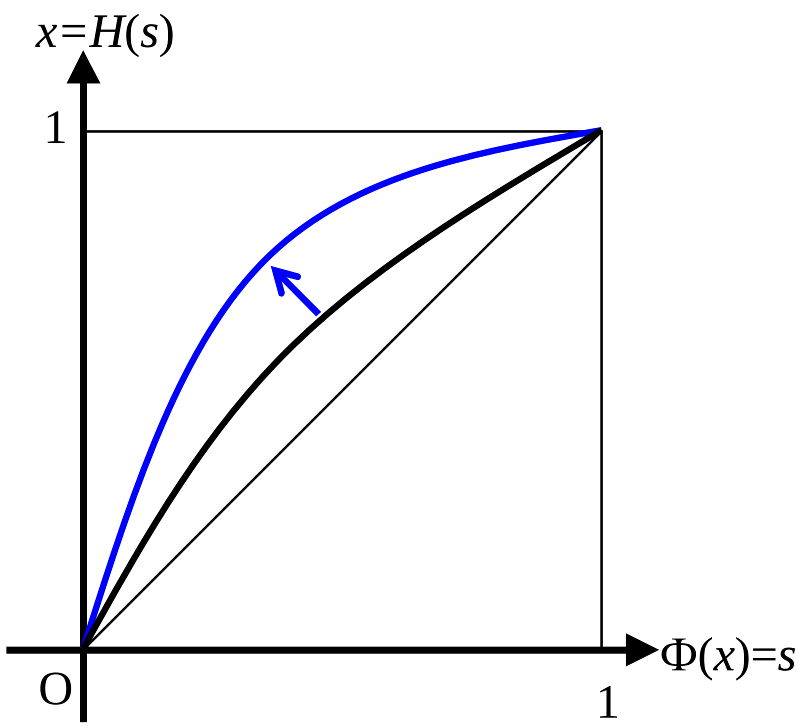
Log-submodularity and Effect of globalization (a higher τ):

Obviously, changing τ = 0 to τ > 0 leads to a greater inequality. How about a small increase in τ?

- $\frac{\partial^2 \ln \hat{h}(s)}{\partial\tau\partial s} < 0$ if $\theta(n) = \theta$. In this case, $\hat{h}(s;g) \equiv e^{-\theta\gamma(s)}\left(1+\frac{\tau\gamma(s)}{(1-\tau)\Gamma^A}\right)^{\frac{(1-\tau)\theta\Gamma^A}{\tau}}$
- $\frac{\partial^2 \ln \hat{h}(s)}{\partial\tau\partial s} < 0$ if $\theta'(n) > 0$ for $n < V\Gamma^A$ and $\theta'(n) < 0$ for $n > V\Gamma^A$

In both cases, a higher τ → more inequality





6.Variable Factor Supply: Effects of Globalization through Factor Mobility or Skill-Biased Technological Change

$V_j = F(K_j, L)$ with $\omega_j F_K(K_j, L) = \rho$

Correlations between *K/L* and TFPs and per capita income

Two Justifications:

- **Factor Mobility:** In a static setting, the rate of return for mobile factors is equalized as they move across borders to seek the highest return.

(If "metropolitan areas," *K* may include not only capital but also labor, with L representing the immobile "land.")

- **Factor Accumulation:** In a dynamic setting, some factors can be accumulated as the representative agent in each country maximizes

$$\int_0^\infty u(C_t)e^{-\rho t}dt \qquad \text{s.t.} \quad Y_t = \left[\int_0^1 \log(X_t(s))ds\right] = C_t + \dot{K}_t$$

Then, the rate of return is equalized in steady state. (In this case, *K* may include not only physical capital but also human capital.)

Condition for Patterns of Trade:

$$\left(\frac{A(n_j)}{A(n_{j+1})}\right)^{\gamma(S_j)} = \frac{\omega_j}{\omega_{j+1}} = \frac{F_K(K_{j+1},L)}{F_K(K_j,L)} < 1 \iff \frac{K_{j+1}}{K_j} > 1 \iff \frac{V_{j+1}}{V_j} > 1.$$

For the *j*-th country which produces $s \in (S_{j-1}, S_j)$,

$$n_j = \Gamma_j V_j = \Gamma_j F(K_j, L);$$

$$\omega_j V_j = \omega_j F(K_j, L) = (S_j - S_{j-1})Y^W.$$

Hence,

$$\frac{Y_{j+1}}{Y_j} = \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \frac{\omega_{j+1} V_{j+1}}{\omega_j V_j} = \frac{V_{j+1}}{V_j}\left(\frac{A(\Gamma_{j+1}V_{j+1})}{A(\Gamma_j V_j)}\right)^{\gamma(S_j)} > 1;$$

For $V_j = F(K_j, L) = ZK_j^{\alpha}$ with $0 < \alpha < 1/(1+\theta(\bullet))$,

Proposition 7 (the *J*-country case): Let S_j be the cumulative share of the J poorest countries in income. Then, $\{S_j\}_{j=0}^{J}$ solves:

$$\frac{Y_{j+1}}{Y_j} = \frac{K_{j+1}}{K_j} = \frac{S_{j+1}-S_j}{S_j-S_{j-1}} = \left(\frac{\omega_{j+1}}{\omega_j}\right)^{\frac{1}{1-\alpha}} = \left(\frac{A\left(Z\left(K_{j+1}\right)^{\alpha}\Gamma(S_j,S_{j+1})\right)}{A\left(Z\left(K_j\right)^{\alpha}\Gamma(S_{j-1},S_j)\right)}\right)^{\frac{\gamma(S_j)}{1-\alpha}} > 1$$

with $S_0 = 0$ and $S_J = 1$, where $\Gamma(S_{j-1},S_j) \equiv \frac{1}{S_j - S_{j-1}}\int_{S_{j-1}}^{S_j}\gamma(s)ds$.

This does not fully characterize the equil. Lorenz curve. We need another condition to pin down the level of *K* (or *Y*). For $A(n) \propto (n)^{\theta}$, this can be rewritten as the 2nd -order difference equation in $\{S_j\}_{j=0}^{J}$, which fully characterize the equilibrium Lorenz curve.

Corollary 3 (the *J*-country case):
Let S_j be the cumulative share of the J poorest countries. Then, $\{S_j\}_{j=0}^{J}$ solves:

$$\frac{S_{j+1}-S_j}{S_j - S_{j-1}} = \left(\frac{\Gamma(S_j,S_{j+1})}{\Gamma(S_{j-1},S_j)}\right)^{\frac{\theta\gamma(S_j)}{1-\alpha-\alpha\theta\gamma(S_j)}} > 1 \text{ with } S_0 = 0 \;\&\; S_J = 1,$$

where $\Gamma(S_{j-1},S_j) \equiv \frac{1}{S_j - S_{j-1}}\int_{S_{j-1}}^{S_j}\gamma(s)ds$.

Calculating the limit:

$$\frac{K_{j+1}}{K_j} = \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{A\left(Z\left(K_{j+1}\right)^{\alpha} \Gamma(S_j, S_{j+1})\right)}{A\left(Z\left(K_j\right)^{\alpha} \Gamma(S_{j-1}, S_j)\right)} \right)^{\gamma(S_j)/(1-\alpha)} > 1$$

with $S_0 = 0$ & $S_J = 1$, where $\Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s) ds$.

As before, by setting $x = j / J$ and $\Delta x = 1/ J$,

$$\text{LHS} = \frac{K_{j+1}}{K_j} = \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = 1 + \frac{\Phi''(x)}{\Phi'(x)} \Delta x + o\left(\left|\Delta x\right|\right)$$

Likewise,

$$\Gamma(S_j, S_{j+1}) = \gamma(\Phi(x)) + \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

$$\Gamma(S_{j-1}, S_j) = \gamma(\Phi(x)) - \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|),$$

$$\left(K(x + \Delta x)\right)^{\alpha} = \left(K(x)\right)^{\alpha} \left(1 + \alpha \frac{\Phi''(x)}{\Phi'(x)} \Delta x \right) + o\left(\left|\Delta x\right|\right),$$

from which

$$A\left(Z\left(K_{j+1}\right)^{\alpha}\Gamma(S_j,S_{j+1})\right)$$
$$=A\left(Z\left(K(x)\right)^{\alpha}\gamma(\Phi(x))\right)\left(1+\theta\left(Z\left(K(x)\right)^{\alpha}\gamma(\Phi(x))\right)\left(\alpha\frac{\Phi''(x)}{\Phi'(x)}+\frac{\gamma'(\Phi(x))\Phi'(x)}{2\gamma(\Phi(x))}\right)\Delta x\right)+o(|\Delta x|)$$
$$A\left(Z\left(K_{j}\right)^{\alpha}\Gamma(S_{j-1},S_{j})\right)$$
$$=A\left(Z\left(K(x)\right)^{\alpha}\gamma(\Phi(x))\right)\left(1-\theta\left(Z\left(K(x)\right)^{\alpha}\gamma(\Phi(x))\right)\left(\frac{\gamma'(\Phi(x))\Phi'(x)}{2\gamma(\Phi(x))}\right)\Delta x\right)+o(|\Delta x|)$$

from which

$$\text{RHS}=\left(\frac{A\left(Z\left(K_{j+1}\right)^{\alpha}\Gamma(S_j,S_{j+1})\right)}{A\left(Z\left(K_{j}\right)^{\alpha}\Gamma(S_{j-1},S_{j})\right)}\right)^{\gamma(S_j)/(1-\alpha)}$$
$$=1+\frac{\gamma(\Phi(x))}{1-\alpha}\theta\left(Z\left(K(x)\right)^{\alpha}\gamma(\Phi(x))\right)\left(\alpha\frac{\Phi''(x)}{\Phi'(x)}+\frac{\gamma'(\Phi(x))\Phi'(x)}{\gamma(\Phi(x))}\right)\Delta x+o(|\Delta x|)$$

Combining these and let $J\to\infty$**,** $\Delta x=1/J\to 0$ yields

$$\frac{\Phi''(x)}{\Phi'(x)}=\frac{\theta\left(Z\left(\overline{K}\Phi'(x)\right)^{\alpha}\gamma(\Phi(x))\right)}{1-\alpha-\alpha\gamma(\Phi(x))\theta\left(Z\left(\overline{K}\Phi'(x)\right)^{\alpha}\gamma(\Phi(x))\right)}\frac{d\gamma(\Phi(x))}{dx}$$

where use has been made of

$$\frac{K'(x)}{K(x)}=\frac{\Phi''(x)}{\Phi'(x)}$$ or $K(x)=\overline{K}\Phi'(x)$, where $\overline{K}$ is the average of K.

By setting $V=Z(\overline{K})^{\alpha}$,

Proposition 8 (Limit Case, *J* → ∞)
The limit equilibrium Lorenz curve in income, Φ, solves:

$$\frac{\Phi''(x)}{\Phi'(x)} = \frac{\theta\left(V\left(\Phi'(x)\right)^{\alpha}\gamma(\Phi(x))\right)}{1-\alpha-\alpha\gamma(\Phi(x))\theta\left(V\left(\Phi'(x)\right)^{\alpha}\gamma(\Phi(x))\right)}\gamma'(\Phi(x))\Phi'(x)$$

with $\Phi(0)=0$ & $\Phi(1)=1$.

Generally, this differential equation has no closed form solution.
For $A(n)\propto(n)^{\theta}$, this can be solved explicitly as follows:

Corollary 4 (Limit Case, *J* → ∞)
The limit equilibrium Lorenz curve, Φ, solves:

$$\frac{\Phi''(x)}{\Phi'(x)} = \frac{\theta}{1-\alpha-\alpha\theta\gamma(\Phi(x))}\frac{d\gamma(\Phi(x))}{dx}$$

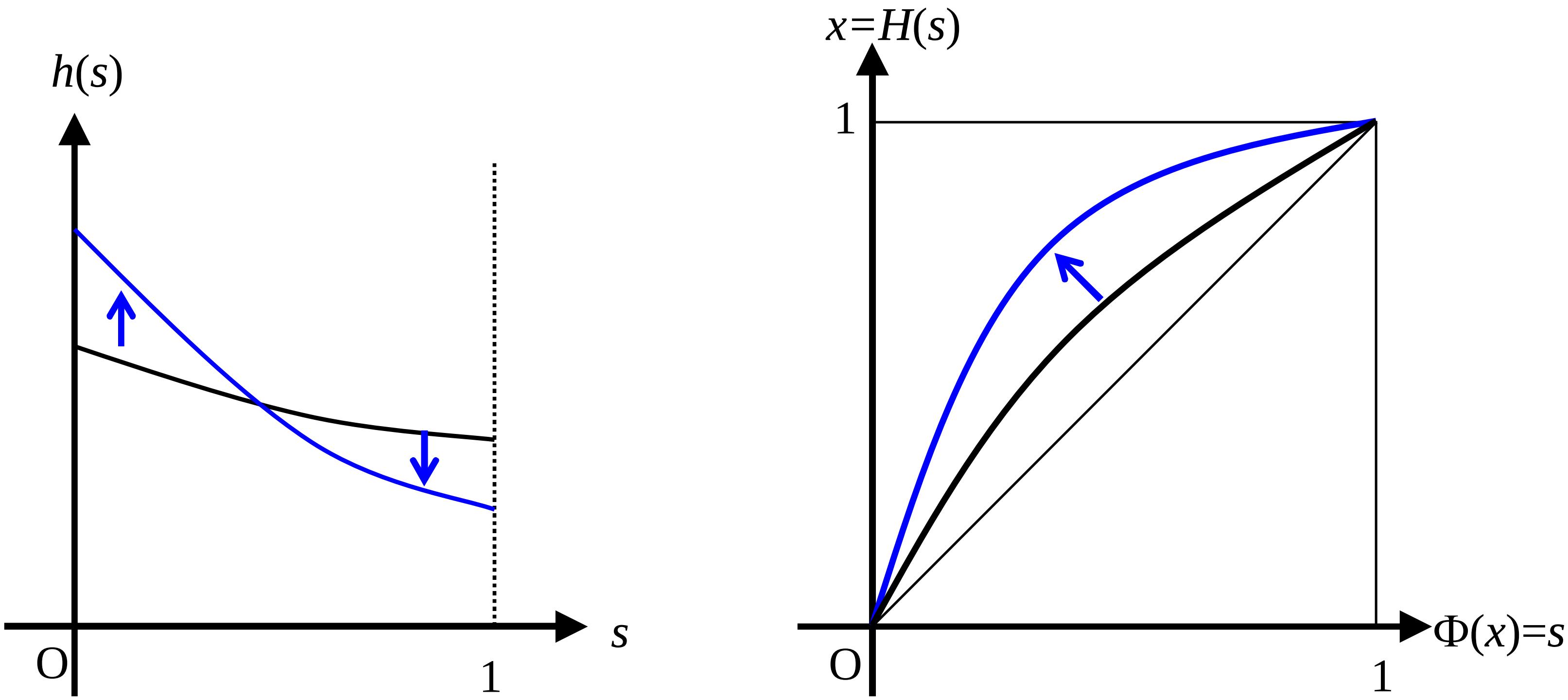
with $\Phi(0)=0$ & $\Phi(1)=1$, whose unique solution is:

$$x = H\left(\Phi(x)\right) \equiv \int_0^{\Phi(x)} h(s)ds, \text{ where } h(s) \equiv \frac{\hat{h}(s)}{\int_0^1 \hat{h}(u)du} \text{ with } \hat{h}(s) \equiv \left(1-\frac{\alpha\theta}{1-\alpha}\gamma(s)\right)^{1/\alpha}$$

NB: Φ is the Lorenz curve in *Y/L* and *K/L*. To obtain the Lorenz curve in TFP,

$$\Phi^{\omega}(x) = \int_0^x \left(\Phi'(u)\right)^{1-\alpha} du \Big/ \left[\int_0^1 \left(\Phi'(u)\right)^{1-\alpha} du\right]$$

Log-Submodularity and Effect of a higher α or a higher θ:



Since $\hat{h}(s) \equiv \left(1 - \dfrac{\alpha\theta}{1-\alpha}\gamma(s)\right)^{1/\alpha}$ is *log-submodular* in α & s (and in θ & s).

7. Some Concluding Remarks:

Symmetry-breaking in general

- Symmetry-breaking due to two-way causality; Even without ex-ante heterogeneity, cross-country dispersion and correlations in per capita income, TFPs, and *K/L* ratios emerge as stable equilibrium patterns due to interaction through trade.
- Some countries become richer (poorer) than others because they trade with poorer (richer) countries. They are *not* independent observations.
- This type of analysis does not say that ex-ante heterogeneity is unimportant. Instead, it says that even small ex-ante heterogeneity could be magnified to create huge ex-post heterogeneity, a possible explanation of Great Divergence and Growth Miracle

This paper in particular

- This paper demonstrates that this type of analysis does not have to be intractable nor lacking in prediction. Equilibrium distribution is *unique, analytically solvable,* varying with parameters in intuitive ways.
- With a finite countries and a continuum of sectors, this model is more compatible with existing quantitative models of trade (Eaton-Kortum, Alvarez-Lucas, etc.)
- A model with many countries can be more tractable than a model with a few countries.